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Automorphisms of cubic Cayley graphs of order $2pq$

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ABSTRACT

In this paper the automorphism groups of connected cubic Cayley graphs of order $2pq$ for distinct odd primes p and q are determined. As an application, all connected cubic non-symmetric Cayley graphs of order $2pq$ are classified and this, together with classifications of connected cubic symmetric graphs and vertex-transitive non-Cayley graphs of order $2pq$ given by the last two authors, completes a classification of connected cubic vertex-transitive graphs of order $2pq$.

1. Introduction

Let G be a permutation group on a set Ω and $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$ and *regular* if G is transitive and semiregular. Throughout this paper a graph means a finite, simple, connected and undirected one. For a graph X, we use $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ to denote its vertex set, edge set and full automorphism group, respectively. For $u, v \in V(X)$, $\{u, v\}$ is the edge incident to u and v in X. An *s-arc* in a graph is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices of the graph such that any two consecutive vertices are adjacent and any three consecutive vertices are distinct. A 1-arc is called an *arc* for short and a 0-arc is a vertex. A graph X is said to be *s-arc-transitive* if $\mathrm{Aut}(X)$ is transitive on the set of s-arcs in X. In particular, 0-arc-transitive means *vertex-transitive*, and 1-arc-transitive means *arc-transitive* or *symmetric*. A symmetric graph X is said to be *s-regular* if the automorphism group $\mathrm{Aut}(X)$ acts regularly on the set of s-arcs in X.

Let G be a finite group and S a subset of G such that $1 \notin S$. The *Cayley digraph* $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set $V(\mathrm{Cay}(G, S)) = G$ and arc set $E(\mathrm{Cay}(G, S)) = \{(g, sg) \mid g \in G, s \in S\}$. If $S = S^{-1}$ then $\mathrm{Cay}(G, S)$, called a *Cayley graph*, is viewed as a graph by identifying two opposite arcs with one edge. It is known that a Cayley digraph $\mathrm{Cay}(G, S)$ is connected if and only if S generates G. Furthermore, $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ is a subgroup of the automorphism group $\mathrm{Aut}(\mathrm{Cay}(G, S))$ of $\mathrm{Cay}(G, S)$. Given a $g \in G$, define the permutation $R(g)$ on G by $x \mapsto xg$, $x \in G$. Then $R(G) = \{R(g) \mid g \in G\}$, called the *right regular representation* of G, is a permutation group isomorphic to G. The Cayley digraph $\mathrm{Cay}(G, S)$ is vertex-transitive because it admits $R(G)$ as a regular subgroup of the automorphism group $\mathrm{Aut}(\mathrm{Cay}(G, S))$. A Cayley digraph $\mathrm{Cay}(G, S)$ is said to be *normal* if $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Xu [30, Proposition 1.5] proved that $\mathrm{Cay}(G, S)$ is normal if and only if $\mathrm{Aut}(\mathrm{Cay}(G, S))_1 = \mathrm{Aut}(G, S)$, where $\mathrm{Aut}(\mathrm{Cay}(G, S))_1$ is the stabilizer of 1 in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. A graph X is isomorphic to a Cayley graph on G if and only if $\mathrm{Aut}(X)$ has a subgroup isomorphic to G, acting regularly on vertices (see [3, Lemma 16.3] or [26, Lemma 4]). For two subsets S and T of G not containing the identity 1, if there is an $\alpha \in \mathrm{Aut}(G)$ such that $S^\alpha = T$ then S and T are said to be *equivalent*, denoted by $S \equiv T$. One may easily show that if $S \equiv T$ then $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$ (graph isomorphic) and then $\mathrm{Cay}(G, S)$ is normal if and only if $\mathrm{Cay}(G, T)$ is normal.

It is well-known that every transitive permutation group of prime degree p is either 2-transitive or solvable with a regular normal Sylow p-subgroup (for example, see [8, Corollary 3.5B]). This implies that a Cayley graph of prime order is normal if

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the graph is neither the empty graph nor the complete graph. Du et al. [11] and Dobson et al., [9] determined the normality of Cayley graphs on groups of order twice a prime and prime square, respectively. Wang et al. [27] obtained all disconnected normal Cayley graphs. Let $\mathrm{Cay}(G, S)$ be a connected cubic Cayley graph on a non-abelian simple group G. Praeger [23] proved that if $N_{\mathrm{Aut}(\mathrm{Cay}(G,S))}(R(G))$ is transitive on edges then the Cayley graph $\mathrm{Cay}(G, S)$ is normal, and Fang et al. [12] proved that the vast majority of connected cubic Cayley graphs on non-abelian simple groups are normal. Recently, Wang and Xu [28] determined the normality of 1-regular tetravalent Cayley graphs on dihedral groups and Feng and Xu [15] proved that every connected tetravalent Cayley graph on a regular p-group is normal when $p \neq 2, 5$. For more results on the normality of Cayley graphs, we refer the reader to [13,16,19,20,30]. The normality of cubic Cayley graphs of order $2p^2$ and $4p$ was determined in [31,32] and in this paper we determine the normality of cubic Cayley graphs of order $2pq$ for distinct odd primes p and q. Furthermore, all cubic non-symmetric Cayley graphs of order $2pq$ are classified, while the classifications of cubic symmetric graphs and vertex-transitive non-Cayley graphs of order $2pq$ were given in [33].

Let $\mathbb{Z}_n$ be the cyclic group of order n, as well as the ring of integers modulo n. Denote by $\mathbb{Z}_n^*$ the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to n and by D_{2n} the dihedral group of order $2n$. For two groups M and N, $N \leq M$ means that N is a subgroup of M and $N < M$ means that N is a proper subgroup of M. By elementary group theory, we know that, up to isomorphism, there are six groups of order $2pq (p > q > 2)$ defined as

$$\begin{aligned}
&G_1(2pq) = \langle a \rangle,\\
&G_2(2pq) = \langle a, b \mid a^{pq} = b^2 = 1, b^{-1}ab = a^{-1} \rangle,\\
&G_3(2pq) = \langle a, b, c \mid a^p = b^q = c^2 = 1, ab = ba, cac = a^{-1}, bc = cb \rangle,\\
&G_4(2pq) = \langle a, b, c \mid a^p = b^q = c^2 = 1, ab = ba, ac = ca, cbc = b^{-1} \rangle,\\
&G_5(2pq) = \langle a, b, c \mid a^p = b^q = c^2 = 1, ac = ca, bc = cb, b^{-1}ab = a^r \rangle,\\
&G_6(2pq) = \langle a, b, c \mid a^p = b^q = c^2 = 1, cac = a^{-1}, bc = cb, b^{-1}ab = a^r \rangle,
\end{aligned} \tag{1}$$

where r is an element of order q in $\mathbb{Z}_p^*$.

2. Preliminaries

For a subgroup H of a group G, denote by $C_G(H)$ the centralizer of H in G and by $N_G(H)$ the normalizer of H in G. Then $C_G(H)$ is normal in $N_G(H)$.

Proposition 2.1 (*[18, I. Theorem 4.5]). The quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group* $\mathrm{Aut}(H)$ *of* H.

The following proposition is a basic fact in permutation group theory.

Proposition 2.2 (*[29, Proposition 4.4]). Every transitive abelian group G on a set Ω is regular and the centralizer $C_{S_\Omega}(G)$ of G in the symmetric group S_Ω is G.*

In view of [7, pp.285, summary], one may extract the following proposition.

Proposition 2.3. *Every maximal subgroup of* $\mathrm{PSL}(2, 7)$ *is isomorphic to $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ or S_4. Let $p = 7, 11$ or 23. All subgroups of* $\mathrm{PGL}(2, p)$ *of order $p(p-1)$ are conjugate and isomorphic to $\mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$, a Frobenius group of degree p.*

The following proposition is known as Burnside's p-q Theorem.

Proposition 2.4 (*[25, Theorem 8.5.3]). Let p and q be primes and let m and n be non-negative integers. Then, any group of order $p^m q^n$ is solvable.*

Let p and q be distinct odd primes. The following result gives the number of solutions of the equation $x^2 + x + 1 = 0$ in $\mathbb{Z}_{pq}$.

Lemma 2.5. *Let $p > q$ be odd primes and $\mathcal{O}_{pq}^3$ the set of solutions of the equation $x^2 + x + 1 = 0$ in $\mathbb{Z}_{pq}$. Then,*

$$|\mathcal{O}_{pq}^3| = \begin{cases} 2 & 3 \mid (p-1) \text{ and } q = 3,\\ 4 & 3 \mid (p-1) \text{ and } 3 \mid (q-1),\\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since $x^3 - 1 = (x-1)(x^2 + x + 1)$, a solution of the equation $x^2 + x + 1 = 0$ must be an element of order 3 in $\mathbb{Z}_{pq}^*$, implying that either $3 \mid (p-1)$ and $q = 3$ or $3 \mid (p-1)$ and $3 \mid (q-1)$. For $3 \mid (p-1)$ and $q = 3$, there are two elements of order 3 in $\mathbb{Z}_{3p}^*$, say x_1 and $x_2 = x_1^2$. Then, $x_i = 1$ in $\mathbb{Z}_3$ for each $i = 1, 2$. Since $(x_i - 1)(x_i^2 + x_i + 1) = x_i^3 - 1 = 0$ in $\mathbb{Z}_{3p}$, it follows that x_1 and x_2 are solutions of $x^2 + x + 1 = 0$ in $\mathbb{Z}_{3p}$. That is $|\mathcal{O}_{3p}^3| = 2$. For $3 \mid (p-1)$ and $3 \mid (q-1)$, a solution k of $x^2+x+1 = 0$ in $\mathbb{Z}_{pq}$ implies that k is an element of order 3 in both $\mathbb{Z}_p^*$ and $\mathbb{Z}_q^*$. Conversely, for every element, say k_1, of order 3 in $\mathbb{Z}_p^*$ and every element, say k_2, of order 3 in $\mathbb{Z}_q^*$, there is a unique element k in $\mathbb{Z}_{pq}$ satisfying the equation $x^2 + x + 1 = 0$ such that $k = k_1 (\mathrm{mod}\ p)$ and $k = k_2 (\mathrm{mod}\ q)$ and this can be easily proved by Eq. (2) in the proof of Lemma 3.1 in [21] which claims that for any $i \in \mathbb{Z}_p$ and $j \in \mathbb{Z}_q$, $|(i + P) \cap (j + Q)| = 1$, where $P = \{sp \mid s \in \mathbb{Z}_q\}$ and $Q = \{sq \mid s \in \mathbb{Z}_p\}$. It follows that $|\mathcal{O}_{pq}^3| = 4$ because there are exactly two elements of order 3 in $\mathbb{Z}_p^*$ and in $\mathbb{Z}_q^*$, respectively. $\square$

Let $p > q$ be primes such that $3 \mid (p-1)$ and $3 \mid (q-1)$. By Lemma 2.5, there are exactly two elements of order 3, say λ and λ^2, in the ring $\mathbb{Z}_{3p}$, and exactly four elements, say λ_1, λ_2, λ_1^2 and λ_2^2, of order 3 satisfying the equation $x^2 + x + 1 = 0$ in $\mathbb{Z}_{pq}$. Define

$$\mathcal{SC}_{6p} = \mathrm{Cay}(D_{6p}, \{b, ba, ba^{-\lambda}\}),$$
$$\mathcal{SC}_{2pq}^1 = \mathrm{Cay}(D_{2pq}, \{b, ba, ba^{-\lambda_1}\}),$$
$$\mathcal{SC}_{2pq}^2 = \mathrm{Cay}(D_{2pq}, \{b, ba, ba^{-\lambda_2}\}),$$

where $D_{2n} = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle$ with $n = 3p$ or pq. It is easy to show that $\mathcal{SC}_{6p}$, $\mathcal{SC}_{2pq}^1$ and $\mathcal{SC}_{2pq}^2$ are independent of the choices λ, λ_1 and λ_2.

Take $H_1 = G_6(2 \cdot 5 \cdot 11) = G_6(110)$ and let $S_1 = \{c, abc, (abc)^{-1}\}$ be a subset of H_1. Take $H_2 = G_6(2 \cdot 11 \cdot 13) = G_6(506)$ and let $S_2 = \{c, ab^3c, (ab^3c)^{-1}\}$ be a subset of H_2. In the groups H_1 and H_2 given in Eq. (1), set $r = 3$ because 3 is an element of order 5 in $\mathbb{Z}_{11}^*$ and an element of order 11 in $\mathbb{Z}_{23}^*$. Define

$$CF_{110} = \mathrm{Cay}(H_1, S_1),$$
$$\mathcal{SC}_{506} = \mathrm{Cay}(H_2, S_2).$$

With the help of software package MAGMA [4], one may easily check $\mathrm{Aut}(CF_{110}) \cong \mathrm{PGL}(2, 11)$ and $\mathrm{Aut}(\mathcal{SC}_{506}) \cong \mathrm{PGL}(2, 23)$. By [5], there is a unique cubic 3-regular graph of order 110 and a unique cubic 4-regular graph of order 506. It follows that these two graphs must be CF_{110} and $\mathcal{SC}_{506}$ because $|\mathrm{PGL}(2, 11)| = 1320$ and $|\mathrm{PGL}(2, 23)| = 12\,144$, of which the first is called Coxeter–Frucht graph (see [6]). Note that PGL(2, 11) and PGL(2, 23) have subgroups of order 110 and 506 by Proposition 2.3 and since these subgroups are Frobenius, they are isomorphic to $G_6(110)$ and $G_6(506)$, respectively. A classification of cubic symmetric graphs of order $2pq$ was given in [33] and one may easily extract those which are Cayley.

Proposition 2.6. *Let $X = \mathrm{Cay}(G, S)$ be a connected cubic symmetric Cayley graph on a group G of order $2pq$, where $p > q$ are odd primes. Then, X is s-regular for $s = 1, 3$ or 4. Furthermore,*

(1) *X is 1-regular if and only if either $q = 3$ and $3 \mid (p-1)$ or $3 \mid (p-1)$ and $3 \mid (q-1)$. If X is 1-regular then it is isomorphic either to $\mathcal{SC}_{6p}$ for $q = 3$ and $3 \mid (p-1)$, or to $\mathcal{SC}_{2pq}^1$ or $\mathcal{SC}_{2pq}^2$ for $3 \mid (p-1)$ and $3 \mid (q-1)$;*
(2) *X is 3-regular if and only if it is isomorphic to CF_{110}. In this case, $G = G_6(110)$, $S \equiv \{c, abc, (abc)^{-1}\}$ (take $r = 3$) and $\mathrm{Aut}(X) \cong \mathrm{PGL}(2, 11)$;*
(3) *X is 4-regular if and only if it is isomorphic to $\mathcal{SC}_{506}$. In this case, $G = G_6(506)$, $S \equiv \{c, ab^3c, (ab^3c)^{-1}\}$ (take $r = 3$) and $\mathrm{Aut}(X) \cong \mathrm{PGL}(2, 23)$.*

Let $X = \mathrm{Cay}(G, S)$ be a Cayley graph on G and $A = \mathrm{Aut}(X)$. It is known that $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ is a subgroup of A. Normal Cayley graphs are those which have the smallest possible automorphism groups.

Proposition 2.7 *([30, Propositions 1.3 and 1.5]). The Cayley graph $X = \mathrm{Cay}(G, S)$ is normal if and only if $A_1 = \mathrm{Aut}(G, S)$ if and only if $A = R(G) \rtimes \mathrm{Aut}(G, S)$, where A_1 is the stabilizer of 1 in A and $R(G)$ is the right regular representation of G.*

By [10, Theorem 1 and Lemma 3.4], we have the following proposition, which can also be deduced from [14,22].

Proposition 2.8. *Let $D_{2n} = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle$ be a dihedral group of order $2n$. A cubic Cayley graph $\mathrm{Cay}(D_{2n}, S)$ on D_{2n} is 1-regular if and only if S is equivalent to $\{b, ba, ba^{-k}\}$ for $n \geq 13$ and $k^2 + k + 1 \equiv 0 (\mathrm{mod}\ n)$. Further, these 1-regular Cayley graphs are normal.*

Let X and Y be two graphs. The *lexicographic product* $X[Y]$ is defined as the graph with vertex set $V(X[Y]) = V(X) \times V(Y)$ such that for any two vertices $u = (x_1, y_1)$ and $v = (x_2, y_2)$ in $V(X[Y])$, u is adjacent to v in $X[Y]$ whenever $\{x_1, x_2\} \in E(X)$ or $x_1 = x_2$ and $\{y_1, y_2\} \in E(Y)$. Denote by K_n the complete graph of order n, C_n the cycle of length n, and $K_{n,n} - nK_2$ the graph by deleting a one factor from the complete bipartite graph $K_{n,n}$ of order $2n$. The following proposition gives all non-normal connected Cayley graphs of valency at most 4 on cyclic groups.

Proposition 2.9 *([2, Corollary 1.3]). All connected Cayley graphs with valency at most 4 on a finite cyclic group are normal, except for $G = \mathbb{Z}_4$ and $X = K_4$, $G = \mathbb{Z}_6$ and $X = K_{3,3}$, $G = \mathbb{Z}_5$ and $X = K_5$, $G = \mathbb{Z}_{2m}$ and $X = C_m[2K_1] (m \geq 3)$, or $G = \mathbb{Z}_{10}$ and $X = K_{5,5} - 5K_2$.*

Given a subset S of a group G with $1 \notin S$, we call S a *CI-subset* of G and $\mathrm{Cay}(G, S)$ a *CI-graph*, if $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$ implies that S and T are equivalent, that is, there exists a $\gamma \in \mathrm{Aut}(G)$ such that $S^\gamma = T$. The following result is a well-known criterion for CI-subset due to Babai [1].

Proposition 2.10. *Let G be a finite group and S a subset of G not containing the identity element 1. Let $X = \mathrm{Cay}(G, S)$ and $A = \mathrm{Aut}(X)$. Then S is a CI-subset of G if and only if for any $\sigma \in S_G$ with $\sigma^{-1}R(G)\sigma \le A$, there exists an $\alpha \in A$ such that $\sigma^{-1}R(G)\sigma = \alpha^{-1}R(G)\alpha$, where S_G denotes the symmetric group on G.*

Qu and Yu [24] investigated the CI-property of Cayley graphs on dihedral groups.

Proposition 2.11 (*[24, Theorem 3.5]*). *Let G be a dihedral group of order $2n$ with n odd and S a subset of G not containing the identity 1. If $|S| \le 3$ then S is a CI-subset.*

3. Automorphism groups of cubic Cayley graphs of order $2pq$

In this section, we shall determine the automorphism groups of cubic Cayley graphs of order $2pq$ for two distinct odd primes p and q. First we prove a lemma which will be used later.

Lemma 3.1. *Let G be a regular subgroup of $\mathrm{Aut}(\mathcal{SC}_{6p})$. Then, $G \cong G_2(6p)$ or $G_6(6p)$. Furthermore, as a Cayley graph on $G_2(6p)$, $\mathcal{SC}_{6p}$ is normal and as a Cayley graph on $G_6(6p)$, $\mathcal{SC}_{6p}$ is non-normal and $\mathcal{SC}_{6p} \cong \mathrm{Cay}(G_6(6p), S)$ with $S \equiv \{c, abc, (abc)^{-1}\}$.*

Proof. Let $X = \mathcal{SC}_{6p}$ and $A = \mathrm{Aut}(X)$. We first claim that A contains regular subgroups isomorphic to $G_6(6p)$. By definition of the graph $\mathcal{SC}_{6p}$, one may assume that $X = \mathrm{Cay}(G_2(6p), S)$, where $G_2(6p) = \langle a, b \mid a^{3p} = b^2 = 1, b^{-1}ab = a^{-1}\rangle$ and $S = \{b, ba, ba^{-k}\}$ with $k^2+k+1 = 0$ in $\mathbb{Z}_{3q}$. Clearly, k has order 3 in $\mathbb{Z}_{3p}^*$. By Proposition 2.8, X is 1-regular and $\mathrm{Cay}(G_2(6p), S)$ is normal. Thus, $A = R(G_2(6p)) \rtimes \langle\alpha\rangle$, where α is an automorphism of order 3 of $G_2(6p)$ induced by $a^\alpha = a^{k^2}$ and $b^\alpha = ba$. Note that $\langle R(a)\rangle \lhd A$. Since each subgroup of $\langle R(a)\rangle$ is characteristic in $\langle R(a)\rangle$, one has $\langle R(a^3)\rangle \lhd A$ and $\langle R(a^p)\rangle \lhd A$. Thus, $\langle R(a^p), \alpha\rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and hence $R(a^p)\alpha$ has order 3. Note that $k^2 + k + 1 = 0$ (in $\mathbb{Z}_{3p}$) implies that $(k, 3) = 1$. It follows $k^2 = 1(\mathrm{mod}\ 3)$. Clearly, $k^2 \ne 1(\mathrm{mod}\ p)$ because $k^2 \ne 1$ in $\mathbb{Z}_{3p}$. Thus, a^{1-k^2} has order p and since $3 \mid (p-1)$, a^{p-1} also has order p, implying that $a^{1-p} = a^{t(1-k^2)}$ for some integer t. Now it is easy to show that $R(ba^t)^{R(a^p)\alpha} = R(ba^t)$. Furthermore, $R(a^3)^{R(a^p)\alpha} = R(a^3)^\alpha = (R(a^3))^{k^2}$ and $R(a^3)^{R(ba^t)} = R(a^3)^{-1}$. Thus, $H = \langle R(a^3), R(a^p)\alpha, R(ba^t)\rangle \cong G_6(6p)$. If the stabilizer H_1 of the identity 1 in H is not trivial, then $H_1 = \mathrm{Aut}(G_2(6p), S) = \langle\alpha\rangle$, forcing $A = H$, a contradiction. Thus, H is regular on $V(X)$, that is, A contains regular subgroups isomorphic to $G_6(6p)$, as claimed.

Let M be an arbitrary regular subgroup of A. If $M \cong G_2(6p)$ then Proposition 2.8 implies that X, as a Cayley graph on $G_2(6p)$, is normal. Now assume $M \ncong G_2(6p)$. Since $|A| = 18p$, one has $A = R(G_2(6p))M$, implying that $|M \cap R(G_2(6p))| = 2p$. Since $G_2(6p)$ has no normal subgroups of order $2p$, M is not normal in A, namely, X, as a Cayley graph on M ($M \ncong G_2(6p)$), is non-normal. Further, since $|M \cap R(G_2(6p))| = 2p$ and $\langle R(a^3)\rangle$ is a normal Sylow p-subgroup of A, one has $\langle R(a^3)\rangle \le M$. As the centralizer $C_A(R(a^3))$ of $R(a^3)$ in A is $\langle R(a)\rangle \cong \mathbb{Z}_{3p}$, one has $C_M(R(a^3)) = M \cap C_A(R(a^3)) = \langle R(a^3)\rangle$. For any given group in Eq. (1), if the centralizer of a Sylow p-subgroup of the group is the Sylow p-subgroup itself then the group must be $G_6(6p)$. It follows that $M \cong G_6(6p)$. Without loss of generality, let $M = G_6(6p) = \langle a, b, c \mid a^p = b^3 = c^2 = 1, cac = a^{-1}, bc = cb, b^{-1}ab = a^r\rangle$ with r an element of order 3 in $\mathbb{Z}_p^*$, and let $X = \mathrm{Cay}(G_6(6p), S)$. Since all involutions of $G_6(6p)$ are conjugate and are contained in $\langle a, c\rangle$, by the connectivity of X, one may assume $S = \{c, y, y^{-1}\}$, where y has order 3 or 6. If y has order 3 then there is a 3-cycle $(1, y, y^{-1}, 1)$ passing through 1, y and y^{-1}, but there is no 3-cycle passing through the vertices $1, c, y$, contrary to the symmetry of X. Thus, y has order 6 and one of y and y^{-1} has form a^ibc, $1 \le i \le p-1$. Since the map $a \mapsto a^i, b \mapsto b, c \mapsto c$ induces an automorphism of $G_6(6p)$, one has $S \equiv \{c, abc, (abc)^{-1}\}$. $\square$

The following is the main result of this section.

Theorem 3.2. *Let $p > q$ be odd primes and let $X = \mathrm{Cay}(G, S)$ be a connected cubic Cayley graph of order $2pq$. Then either $\mathrm{Aut}(X) = R(G) \rtimes \mathrm{Aut}(G, S)$ or one of the following holds:*

(1) *$G = G_6(6p)$ with $3 \mid (p-1)$, $S \equiv \{c, abc, (abc)^{-1}\}$ and $\mathrm{Aut}(X) \cong G_6(6p)\mathbb{Z}_3$;*
(2) *$G = G_6(110)$, $S \equiv \{c, abc, (abc)^{-1}\}$ (take $r = 3$) and $\mathrm{Aut}(X) \cong \mathrm{PGL}(2, 11)$;*
(3) *$G = G_6(506)$, $S \equiv \{c, ab^3c, (ab^3c)^{-1}\}$ (take $r = 3$) and $\mathrm{Aut}(X) \cong \mathrm{PGL}(2, 23)$;*
(4) *$G = G_6(42)$, $S \equiv \{c, ab, (ab)^{-1}\}$ and $\mathrm{Aut}(X) \cong \mathrm{PGL}(2, 7)$.*

Proof. Let $A = \mathrm{Aut}(X)$. Assume that $\mathrm{Aut}(X) > R(G) \rtimes \mathrm{Aut}(G, S)$, that is, $R(G)$ is not normal in A. We deal with two cases depending on the symmetry of X.

Case I: X is symmetric.

By Proposition 2.6, X is isomorphic to CF_{110}, $\mathcal{SC}_{506}$, $\mathcal{SC}_{6p}$, $\mathcal{SC}^1_{2pq}$ or $\mathcal{SC}^2_{2pq}$. If $X \cong \mathcal{SC}_{6p}$, then by Lemma 3.1, $G \cong G_6(6p)$ and $S \equiv \{c, abc, (abc)^{-1}\}$, that is the case (1) in the theorem. Assume $X \cong CF_{110}$. Then $\mathrm{Aut}(X) \cong \mathrm{PGL}(2, 11)$, and by Proposition 2.3, one may assume that $X = \mathrm{Cay}(G_6(110), S)$, where $G_6(110) = \langle a, b, c \mid a^{11} = b^5 = c^2 = 1, cac =$

$a^{-1}, bc = cb, b^{-1}ab = a^3\rangle$ (take $r = 3$). For any $\sigma \in S_G$ such that $\sigma^{-1}R(G_6(110))\sigma \leq \mathrm{Aut}(X)$, again by Proposition 2.3, $\sigma^{-1}R(G_6(110))\sigma$ and $R(G_6(110))$ are conjugate in $\mathrm{Aut}(X)$ because they have the same order $11 \cdot 10$, and by Proposition 2.10, X is a CI-graph. This implies that $S \equiv \{c, cab, (cab)^{-1}\}$ by Proposition 2.6, which is the Case 2 in the theorem. Similarly, if $X \cong \mathcal{SC}_{506}$ then we have the Case 3 in the theorem.

Assume $X \cong \mathcal{SC}^1_{2pq}$ or $\mathcal{SC}^2_{2pq}$. By Proposition 2.6, one has $3 \mid (p-1)$ and $3 \mid (q-1)$, and X is a 1-regular Cayley graph on the dihedral group $G_2(2pq) = \langle a, b \mid a^{pq} = b^2 = 1, b^{-1}ab = a^{-1}\rangle$. The Cayley graph is normal by Proposition 2.8. By the 1-regularity of X, $A \cong R(G_2(2pq)) \rtimes \mathbb{Z}_3$ and it is easy to show that $R(G_2(2pq))$ is the unique regular subgroup of A because $p > q > 3$, implying that X cannot be a Cayley graph on $G_i(2pq)$ for $i = 1, 3, 4, 5$ or 6.

Case II: X is non-symmetric.

In this case, the stabilizer A_v of $v \in V(X)$ in A is a 2-group and hence $|A| = 2^\ell \cdot p \cdot q$ with $\ell \geq 2$. We claim that A has no normal 2-subgroups. Suppose to the contrary that H is a normal 2-subgroup of A. Let X_H be the quotient graph of X relative to H, that is, the graph with vertices the orbits of H in $V(X)$ and with two orbits adjacent if there is an edge in X between those two orbits. Let K be the kernel of A acting on $V(X_H)$. Then, $H \leq K$ and A/K is transitive on $V(X_H)$. Since $|V(X)| = 2pq$, every orbit of H in $V(X)$ has length 2, implying $|V(X_H)| = pq$. As X has valency 3 and $H \leq K$, X_H has valency 2 or 3, and since pq is odd, X_H has valency 2. By the connectivity of X, X_H is a cycle of length pq, say $V(X_H) = \{B_0, B_1, \ldots, B_{pq-1}\}$, where B_i is adjacent to B_{i+1} for each $i \in \mathbb{Z}_{pq}$. If there is no edge in each B_i, then one may assume that each vertex in B_1 is adjacent to one vertex in B_0 and two vertices in B_2. By the transitivity of A/K on $V(X_H)$, the length of the cycle X_H must be even, contrary to the fact that pq is odd. If there is an edge in some B_{i_0} then there is an edge in each B_i, $0 \leq i \leq pq-1$, because of the transitivity of A/K on $V(X_H)$. Since K fixes each orbit of H setwise, the stabilizer K_v of $v \in V(X)$ in K fixes every neighbor of v in X. The connectivity of X gives $K_v = 1$, forcing $K = H \cong \mathbb{Z}_2$. Since $A/K = A/H \leq \mathrm{Aut}(X_H) \cong D_{2pq}$, one has $|A| \leq 4pq$ and hence $|A : R(G)| \leq 2$, implying $R(G) \trianglelefteq A$, a contradiction. Thus, the claim is true, that is, A has no normal 2-subgroups. In what follows we assume that N is a minimal normal subgroup of A. Then N is $\mathbb{Z}_p$ or $\mathbb{Z}_q$, or a non-abelian simple group because $|N| \mid 2^\ell \cdot p \cdot q$.

Assume that N is solvable. Then $N \cong \mathbb{Z}_p$ or $\mathbb{Z}_q$. By Proposition 2.4, A/N is solvable. Let $C = C_A(N)$. By Proposition 2.1, $A/C \leq \mathrm{Aut}(N) \cong \mathbb{Z}_{p-1}$ or $\mathbb{Z}_{q-1}$. Clearly, $N \leq C$. There are two subcases: $N = C$ and $N < C$, that is, N is a proper subgroup of C.

Suppose $N = C$. Then $A/N \leq \mathrm{Aut}(N) \cong \mathbb{Z}_{p-1}$ or $\mathbb{Z}_{q-1}$. Since $p > q > 2$, one has $N \cong \mathbb{Z}_p$ and $A/N \leq \mathbb{Z}_{p-1}$. Let X_N be the quotient graph of X relative to the orbits of N, and K the kernel of A acting on $V(X_N)$. Then, $N \leq K$ and A/K is transitive on $V(X_N)$. Since N is normal in A, X_N has valency at most 3, and since $N \cong \mathbb{Z}_p$, one has $|V(X_N)| = 2q > 1$, implying that X_N has valency 2 or 3. If X_N has valency 3 then K has trivial stabilizers and hence $K = N$. By Proposition 2.2, A/N is regular on $V(X_N)$ because $A/N \leq \mathbb{Z}_{p-1}$. It follows that $|A| = 2pq$, forcing $R(G) \trianglelefteq A$, a contradiction. If X_N has valency 2 then X_N is a cycle of length $2q$ because of the connectivity of X. Let $V(X_N) = \{B_0, B_1, \ldots, B_{2q-1}\}$ with B_i adjacent to B_{i+1} for every $i \in \mathbb{Z}_{2q-1}$. If there is an edge of X in each B_i then the induced subgraph $\langle B_i\rangle$ of B_i in X must be a cycle of length p because $|B_i| = p$ is odd. In this case, X_N has valency 1, a contradiction. Thus, there is no edge in each B_i and one may assume that each vertex in B_1 connects one vertex in B_0 and two vertices in B_2. It follows that the induced subgraph $\langle B_0 \cup B_1\rangle$ of $B_0 \cup B_1$ in X is a perfect matching and the induced subgraph $\langle B_1 \cup B_2\rangle$ of $B_1 \cup B_2$ in X is a cycle of length $2p$ because $|B_1| = p$ is odd. Thus, A/K is not arc-transitive on X_N, and hence $A/K < \mathrm{Aut}(X/N) \cong D_{4q}$, implying $|A/K| = 2q$ by the vertex-transitivity of A/K on $V(X_N)$. Further, K acts faithfully on B_1 and $K < \mathrm{Aut}(\langle B_1 \cup B_2\rangle) \cong D_{4p}$. It follows that $|K| \leq 2p$ and hence $|A| \leq 4pq$. Thus, $R(G) \trianglelefteq A$ because $|A : R(G)| \leq 2$, a contradiction.

Suppose $N < C$. Take a minimal normal subgroup of A/N, say M/N, in C/N. Since A/N is solvable, M/N is elementary abelian. It follows that either M/N is a 2-group, or $M/N \cong \mathbb{Z}_q$ or $\mathbb{Z}_p$. For the former, one has $|M| = 2^s \cdot p$ or $2^s \cdot q$ for some integer $s \geq 1$. Since $M \leq C$, a Sylow 2-subgroup of M is characteristic in M, and hence normal in A because $M \trianglelefteq A$. This is impossible because A has no normal 2-subgroups. Thus, $M/N \cong \mathbb{Z}_q$ or $\mathbb{Z}_p$, and hence $M \cong \mathbb{Z}_{pq}$ because $M \leq C$. Clearly, $M \leq C_A(M)$. If $M = C_A(M)$ then, by Proposition 2.1, $A/M \leq \mathrm{Aut}(M) \cong \mathbb{Z}_{p-1} \times \mathbb{Z}_{q-1}$. Since $M \trianglelefteq A$, one has $M \leq R(G)$, implying $R(G)/M \trianglelefteq A/M$, that is, $R(G) \trianglelefteq A$, a contradiction. If $M < C_A(M)$ then $C_A(M)/M$ must be a 2-group. It follows that $C_A(M) = M \times Q$, where Q is a Sylow 2-subgroup of $C_A(M)$. Thus, Q is characteristic in $C_A(M)$ and normal in A because $C_A(M) \trianglelefteq A$, contrary to the fact that A has no normal 2-subgroups.

Assume that N is insolvable. Since $|N| \mid 2^\ell \cdot p \cdot q$ and $p > q > 2$, N must be non-abelian simple and by [17, pp. 12–14], N is one of the following groups:

$$A_5, A_6, \mathrm{PSL}(2,7), \mathrm{PSL}(2,8), \mathrm{PSL}(2,17), \mathrm{PSL}(3,3), \mathrm{PSU}(3,3) \quad \text{and} \quad \mathrm{PSU}(4,2).$$

Since $p^2 \nmid |N|$ and $q^2 \nmid |N|$, by checking the orders of the above groups, one has $N = A_5$ or $\mathrm{PSL}(2,7)$. Let $C = C_A(N)$. Then $N \cap C = 1$ because N is simple. It follows that either C is a 2-subgroup or $C = 1$. Thus, $C = 1$ because A has no normal 2-subgroups, and by Proposition 2.1, one has $A \leq \mathrm{Aut}(N)$. If $N = A_5$ then $A = A_5$ or S_5. However, both S_5 and A_5 have no subgroups of order 30, implying that X is a non-Cayley graph, a contradiction. It follows that $N = \mathrm{PSL}(2,7)$ and $A \leq \mathrm{Aut}(N) \cong \mathrm{PGL}(2,7)$. Since X is a Cayley graph, A contains a regular subgroup of order 42 and by Proposition 2.3, $\mathrm{PSL}(2,7)$ has no subgroups of order 42, implying $A = \mathrm{PGL}(2,7)$. By Proposition 2.3, every subgroup of order 42 in $\mathrm{PGL}(2,7)$ is conjugate to $G_6(42)$. Without loss of generality, let $G = G_6(42) = \langle a, b, c \mid a^7 = b^3 = c^2 = 1, cac = a^{-1}, bc = cb, b^{-1}ab = a^2\rangle$. Clearly, all involutions in G are conjugate and hence one may assume $c \in S$. Note that the centralizer of c in G has order 6 and so there are seven involutions in G, of which all are contained in $\langle a, c\rangle$. Since S generates G, $S = \{c, y, y^{-1}\}$, where y has order 3 or 6. If y has order 6 then one of y and y^{-1} has the form a^ibc, $1 \le i \le 6$, and since the map $a \mapsto a^i$, $b \mapsto b$, $c \mapsto c$ induces an automorphism of $G_6(6p)$, one may further assume $S = \{c, abc, (abc)^{-1}\}$. If y has order 3, one of y and y^{-1} has the form a^ib, $1 \le i \le 6$, and similarly one may assume $S = \{c, ab, (ab)^{-1}\}$. With the help of computer software package MAGMA [4], $|\mathrm{Aut}(X)| = 3 \cdot 42$ for $S = \{c, abc, (abc)^{-1}\}$ and $\mathrm{Aut}(X) \cong \mathrm{PGL}(2, 7)$ for $S = \{c, ab, (ab)^{-1}\}$. For the former, X is arc-transitive, a contradiction, and for the latter, X is not normal because $\mathrm{PGL}(2, 7)$ has no normal subgroup of order 42, which is the Case (4) in the theorem. $\square$

4. Cubic non-symmetric Cayley graphs of order 2*pq*

Let $p > q$ be odd primes. In this section we shall classify connected cubic non-symmetric Cayley graphs of order $2pq$. For $x \in \mathbb{Z}_{pq}^*$, denote x^{-1} the inverse of x in the multiple group $\mathbb{Z}_{pq}^*$. Let $\mathcal{O}_{pq}^3$ be the set of solutions of the equation $x^2 + x + 1 = 0$ in $\mathbb{Z}_{pq}$. By Lemma 2.5, $|\mathcal{O}_{pq}^3| = 2$ for $3 \mid (p-1)$ and $q = 3$, $|\mathcal{O}_{pq}^3| = 4$ for $3 \mid (p-1)$ and $3 \mid (q-1)$, and $|\mathcal{O}_{pq}^3| = 0$ otherwise. There are exactly three involutions in $\mathbb{Z}_{pq}^*$, denoted by λ_1, λ_2 and λ_3. Set

$$\begin{aligned} \Lambda &= \{\lambda_1, \lambda_2, \lambda_3\}, \\ \Theta &= \mathbb{Z}_{pq} - \{\{0, 1, 2^{-1}, \lambda_1, \lambda_2, \lambda_3, 1-\lambda_1, 1-\lambda_2, 1-\lambda_3\} \cup \{-\mathcal{O}_{pq}^3\}\}. \end{aligned} \tag{2}$$

Now we introduce some cubic non-symmetric Cayley graphs of order $2pq$.

Example 4.1. Let $G = \langle a, b \mid a^{pq} = b^2 = 1, b^{-1}ab = a^{-1}\rangle$. Define

$$\begin{aligned} \mathcal{C}_{2pq}^1 &= \mathrm{Cay}(G, \{b, a, a^{-1}\}), \\ \mathcal{C}_{2pq}^{2,\lambda} &= \mathrm{Cay}(G, \{b, ba, ba^{\lambda}\}), \quad \lambda \in \Lambda \\ \mathcal{C}_{2pq}^{3,\mu} &= \mathrm{Cay}(G, \{b, ba, ba^{\mu}\}), \quad \mu \in \Theta. \end{aligned}$$

Then we have the following:

(1) For each $k \in \{2^{-1}, \lambda, 1-\lambda \mid \lambda \in \Lambda\}$, the Cayley graph $\mathrm{Cay}(G, \{b, ba, ba^k\})$ is isomorphic to one of $\mathcal{C}_{2pq}^{2,\lambda}$, $\lambda \in \Lambda$.

(2) The graphs $\mathcal{C}_{2pq}^1$, $\mathcal{C}_{2pq}^{2,\lambda}$ and $\mathcal{C}_{2pq}^{3,\mu}$ are connected cubic non-symmetric Cayley graphs of order $2pq$. Moreover, $\mathrm{Aut}(\mathcal{C}_{2pq}^1) \cong \mathrm{Aut}(\mathcal{C}_{2pq}^{2,\lambda}) \cong G \rtimes \mathbb{Z}_2$ and $\mathrm{Aut}(\mathcal{C}_{2pq}^{3,\mu}) \cong G$.

(3) The graphs $\mathcal{C}_{2pq}^1$, $\mathcal{C}_{2pq}^{2,\lambda}$, $\lambda \in \Lambda$, are pairwise non-isomorphic.

(4) For $\mu_1, \mu_2 \in \Theta$, $\mathcal{C}_{2pq}^{3,\mu_1} \cong \mathcal{C}_{2pq}^{3,\mu_2}$ if and only if one of the following holds in the ring $\mathbb{Z}_{pq}$: $\mu_1\mu_2 = 1$, $\mu_1 + \mu_2 = 1$, $\mu_1(1-\mu_2) = 1$, $\mu_2(1-\mu_1) = 1$, $\mu_1 + \mu_2 - \mu_1\mu_2 = 0$.

Proof. The automorphism of G induced by $b \mapsto ba$ and $a \mapsto a^{-1}$ maps $\{b, ba, ba^{\lambda_i}\}$ to $\{b, ba, ba^{1-\lambda_i}\}$, and the automorphism of G induced by $b \mapsto ba^{2^{-1}}$ and $a \mapsto a^{-2^{-1}}$ maps $\{b, ba, ba^{-1}\}$ to $\{b, ba, ba^{2^{-1}}\}$. Since one of λ_1, λ_2 and λ_3 must be -1, (1) follows.

Set $S_1 = \{b, a, a^{-1}\}$, $S_2 = \{b, ba, ba^{\lambda}\}$ and $S_3 = \{b, ba, ba^{\mu}\}$, where $\lambda \in \Lambda$ and $\mu \in \Theta$. Since $\langle S_i\rangle = G$ $(1 \le i \le 3)$, the graphs $\mathcal{C}_{2pq}^1$, $\mathcal{C}_{2pq}^{2,\lambda}$ and $\mathcal{C}_{2pq}^{3,\mu}$ are connected cubic Cayley graphs, which are normal by Theorem 3.2. Thus, $\mathrm{Aut}(\mathrm{Cay}(G, S_i)) = R(G) \rtimes \mathrm{Aut}(G, S_i)$. To prove (2), it suffices to show that $\mathrm{Aut}(G, S_1) \cong \mathrm{Aut}(G, S_2) \cong \mathbb{Z}_2$ and $\mathrm{Aut}(G, S_3) = 1$. Since S_1 contains only one involution, $\mathrm{Aut}(G, S_1) \le \mathbb{Z}_2$ and since the automorphism of G induced by $b \mapsto b$ and $a \mapsto a^{-1}$ fixes S_1, one has $\mathrm{Aut}(G, S_1) \cong \mathbb{Z}_2$. Let $S = \{b, ba, ba^k\}$ with $k \ne 0, 1$. It is easy to check that $3 \mid |\mathrm{Aut}(G, S)|$ if and only if there is $\alpha \in \mathrm{Aut}(G)$ such that α permutes $\{b, ba, ba^k\}$ cyclically if and only if $-k \in \mathcal{O}_{pq}^3$. It follows that $\mathrm{Aut}(G, S_2) \cong \mathbb{Z}_2$ because the map $a \mapsto a^{\lambda}$ and $b \mapsto b$ induces an automorphism of G of order 2 that fixes S_2. Furthermore, $\mathrm{Aut}(G, S) \cong \mathbb{Z}_2$ if and only if there is an element of order 2 in $\mathrm{Aut}(G)$ that fixes one element in S and interchanges the other two in S if and only if one of the following holds in $\mathbb{Z}_{pq}$: $k^2 = 1$, $k(k-2) = 0$ and $2k - 1 = 0$. Note that the map $x \mapsto 1 - x$ is a bijection between the solution sets of the equations $k^2 = 1$ and $k(k-2) = 0$ in $\mathbb{Z}_{pq}$. Thus, $\mathrm{Aut}(G, S) = 1$ if and only if $k \in \Theta$, which implies that $\mathrm{Aut}(G, S_3) = 1$.

By Proposition 2.11, any 3-subset of G not containing the identity is a CI-subset. Thus, for each $\lambda \in \Lambda$ we have $\mathcal{C}_{2pq}^1 \ncong \mathcal{C}_{2pq}^{2,\lambda}$ because S_1 contains only one involution and S_2 consists of involutions. Also, it is easy to check that $\{b, ba, ba^{\lambda_1}\}$, $\{b, ba, ba^{\lambda_2}\}$ and $\{b, ba, ba^{\lambda_3}\}$ are pairwise non-equivalent. Thus, $\mathcal{C}_{2pq}^1$, $\mathcal{C}_{2pq}^{2,\lambda}$, $\lambda \in \Lambda$, are pairwise non-isomorphic.

Note that $\mathrm{Cay}(G, \{b, ba, ba^{\mu_1}\}) \cong \mathrm{Cay}(G, \{b, ba, ba^{\mu_2}\})$ if and only if there exists $\beta \in \mathrm{Aut}(G)$ such that $\{b, ba, ba^{\mu_1}\}^{\beta} = \{b, ba, ba^{\mu_2}\}$. This is true if and only if one of the following holds in the ring $\mathbb{Z}_{pq}$: $\mu_1\mu_2 = 1$, $\mu_1 + \mu_2 = 1$, $\mu_1(1-\mu_2) = 1$, $\mu_2(1-\mu_1) = 1$, $\mu_1 + \mu_2 - \mu_1\mu_2 = 0$. The proof is straightforward. For example, there exists an automorphism of G that maps ba^{μ_1} to ba^{μ_2} and interchanges b and ba if and only if $\mu_1\mu_2 = 1$. $\square$

Example 4.2. Let $G = \langle a, b, c \mid a^p = b^q = c^2 = 1, ab = ba, cac = a^{-1}, bc = cb\rangle$. Define

$$\mathcal{C}_{2pq}^4 = \mathrm{Cay}(G, \{c, ab, (ab)^{-1}\}).$$

Then $\mathcal{C}_{2pq}^4$ is a cubic non-symmetric Cayley graph and $\mathrm{Aut}(\mathcal{C}_{2pq}^4) \cong G \rtimes \mathbb{Z}_2$.

Proof. Set $S = \{c, ab, (ab)^{-1}\}$. One may easily show that $\mathrm{Aut}(G, S) = \langle\alpha\rangle \cong \mathbb{Z}_2$, where α is the automorphism of G induced by $a \mapsto a^{-1}$, $b \mapsto b^{-1}$ and $c \mapsto c$. By Theorem 3.2, $\mathrm{Cay}(G, S)$ is normal, and hence $\mathrm{Aut}(\mathcal{C}^4_{2pq}) \cong R(G) \rtimes \mathbb{Z}_2$, implying that $\mathcal{C}^4_{2pq}$ is a cubic non-symmetric Cayley graph. □

Example 4.3. Let $G = \langle a, b, c \mid a^p = b^q = c^2 = 1, cac = a^{-1}, bc = cb, b^{-1}ab = a^r\rangle$ where r is an element of order q in $\mathbb{Z}^*_p$, and set $r = 3$ for $(p, q) = (11, 5)$ or $(23, 11)$. Define

$$\mathcal{C}^{5,\xi}_{2pq} = \mathrm{Cay}(G, \{c, ab^{\xi}, (ab^{\xi})^{-1}\}),$$
$$\mathcal{C}^{6,\varsigma}_{2pq} = \mathrm{Cay}(G, \{c, ab^{\varsigma}c, (ab^{\varsigma}c)^{-1}\}),$$

where $1 \le \xi, \varsigma \le \frac{q-1}{2}$. Then we have the following:

(1) The graph $\mathcal{C}^{5,\xi}_{2pq}$ is non-symmetric. Furthermore,

$$\mathrm{Aut}(\mathcal{C}^{5,\xi}_{2pq}) = \begin{cases} R(G) & (p, q) \ne (7, 3) \\ \mathrm{PGL}(2, 7) & (p, q) = (7, 3); \end{cases}$$

(2) The graph $\mathcal{C}^{6,\varsigma}_{2pq}$ is non-symmetric if and only if $p > q > 3$ and $(p, q, \varsigma) \ne (11, 5, 1), (23, 11, 3)$. If $\mathcal{C}^{6,\varsigma}_{2pq}$ is non-symmetric then $\mathrm{Aut}(\mathcal{C}^{6,\varsigma}_{2pq}) = R(G)$;

(3) The graphs $\mathcal{C}^{5,\xi}_{2pq}$ and $\mathcal{C}^{6,\varsigma}_{2pq}$, $1 \le \xi, \varsigma \le \frac{q-1}{2}$, are pairwise non-isomorphic.

Proof. Suppose there is an $\alpha \in \mathrm{Aut}(G)$ such that $(ab^{k_1})^{\alpha} = (a^{\delta}b^{k_2})^{-1}$, where $1 \le k_1, k_2 \le \frac{q-1}{2}$ and $\delta = \pm 1$. Since $\langle a\rangle$ is characteristic in G, one has $a^{\alpha} = a^t$ for some $t \in \mathbb{Z}^*_p$. Clearly, $(ab^{k_1})^{-1}a(ab^{k_1}) = a^{r^{k_1}}$. It follows that $(a^{\delta}b^{k_2})a^t(a^{\delta}b^{k_2})^{-1} = a^{tr^{k_1}}$, namely, $a^{tr^{-k_2}} = a^{tr^{k_1}}$. Then $r^{k_1+k_2} = 1 (\mathrm{mod}\ p)$ and hence $q \mid (k_1 + k_2)$ because r is an element of order q in $\mathbb{Z}^*_p$. This is impossible because $2 \le k_1 + k_2 < q$. Thus, there is no $\alpha \in \mathrm{Aut}(G)$ such that $(ab^{k_1})^{\alpha} = (a^{\delta}b^{k_2})^{-1}$ for any $1 \le k_1, k_2 \le \frac{q-1}{2}$ and $\delta = \pm 1$.

For each $1 \le \xi, \varsigma \le \frac{q-1}{2}$, ab^{ξ} has order q and $ab^{\varsigma}c$ has order $2q$. This implies that $\{c, ab^{\xi}, (ab^{\xi})^{-1}\} \not\equiv \{c, ab^{\varsigma}c, (ab^{\varsigma}c)^{-1}\}$. Let $S = \{c, ab^{\xi}, (ab^{\xi})^{-1}\}$ and $A = \mathrm{Aut}(\mathcal{C}^{5,\xi}_{2pq})$. Note that if $q = 3$ then $\xi = 1$. By Theorem 3.2, if $(p, q) \ne (7, 3)$ then $\mathcal{C}^{5,\xi}_{2pq} = \mathrm{Cay}(G, S)$ is normal and if $(p, q) = (7, 3)$ then $\mathrm{Aut}(\mathcal{C}^{5,\xi}_{42}) = \mathrm{Aut}(\mathcal{C}^{5,1}_{42}) \cong \mathrm{PGL}(2, 7)$. Since $|\mathrm{PGL}(2, 7)| = 42 \times 8$, $\mathcal{C}^{5,1}_{42}$ is non-symmetric. Assume $(p, q) \ne (7, 3)$. By Proposition 2.7, $A = R(G)\mathrm{Aut}(G, S)$. Let $\alpha \in \mathrm{Aut}(G, S)$. As c is the unique involution in S, α fixes c. By the first paragraph, there is no α in $\mathrm{Aut}(G)$ interchanging ab^{ξ} and $(ab^{\xi})^{-1}$. Thus, $(ab^{\xi})^{\alpha} = (ab^{\xi})$ and since $G = \langle c, ab^{\xi}\rangle$, one has $\alpha = 1$. This implies that $A = R(G)$ and hence $\mathcal{C}^{5,\xi}_{2pq}$ is non-symmetric.

By Theorem 3.2, $\mathcal{C}^{6,1}_{2\cdot 3\cdot p}$, $\mathcal{C}^{6,1}_{2\cdot 5\cdot 11}$ and $\mathcal{C}^{6,3}_{2\cdot 23\cdot 11}$ are symmetric. Note that if $q = 3$ then $\varsigma = 1$. It follows that if $\mathcal{C}^{6,\varsigma}_{2pq}$ is non-symmetric then $p > q > 3$ and $(p, q, \varsigma) \ne (11, 5, 1), (23, 11, 3)$. Conversely, assume $p > q > 3$ and $(p, q, \varsigma) \ne (11, 5, 1), (23, 11, 3)$. To finish the proof of (2), it suffices to show that $\mathrm{Aut}(\mathcal{C}^{6,\varsigma}_{2pq}) = R(G)$. If $(p, q) = (11, 5)$, with the help of computer software package MAGMA [4], one may compute that $|\mathrm{Aut}(\mathcal{C}^{6,\varsigma}_{2\cdot 5\cdot 11})| = 110$ for $\varsigma = 2$ and hence $\mathrm{Aut}(\mathcal{C}^{6,2}_{2\cdot 5\cdot 11}) = R(G)$. Similarly, if $(p, q) = (23, 11)$ then $\mathrm{Aut}(\mathcal{C}^{6,\varsigma}_{2\cdot 11\cdot 23}) = R(G)$ for $\varsigma = 1, 2, 4, 5$. Thus, one may assume that $(p, q) \ne (11, 5), (23, 11)$. Since $p > q > 3$, by Theorem 3.2, $\mathcal{C}^{6,\varsigma}_{2pq}$ is normal. Let $S = \{c, ab^{\varsigma}c, (ab^{\varsigma}c)^{-1}\}$ and $A = \mathrm{Aut}(\mathcal{C}^{6,\varsigma}_{2pq})$. By Proposition 2.7, $A = R(G)\mathrm{Aut}(G, S)$. Let $\beta \in \mathrm{Aut}(G, S)$. Clearly, $c^{\beta} = c$. Suppose that $(ab^{\varsigma}c)^{\beta} = (ab^{\varsigma}c)^{-1}$. Then, $(ab^{\varsigma})^{\beta} = (a^{-1}b^{\varsigma})^{-1}$ which is impossible because of the argument in the first paragraph. Thus, $(ab^{\varsigma}c)^{\beta} = ab^{\varsigma}c$ and hence $\beta = 1$ because $G = \langle c, ab^{\varsigma}c\rangle$, which implies that $\mathrm{Aut}(\mathcal{C}^{6,\varsigma}_{2pq}) = R(G)$, as required.

If $q = 3$ then $\xi = \varsigma = 1$. To prove (3), one may assume that $p > q > 3$ and $(p, q, \varsigma) \ne (11, 5, 1), (23, 11, 3)$. Thus, $\mathrm{Aut}(\mathcal{C}^{5,\xi}_{2pq}) = \mathrm{Aut}(\mathcal{C}^{6,\varsigma}_{2pq}) = R(G)$. By Proposition 2.10, $\mathcal{C}^{5,\xi}_{2pq}$ and $\mathcal{C}^{6,\varsigma}_{2pq}$ are CI-graphs. Since $\{c, ab^{\xi}, (ab^{\xi})^{-1}\} \not\equiv \{c, ab^{\varsigma}c, (ab^{\varsigma}c)^{-1}\}$, it suffices to show that $\mathcal{C}^{5,\xi}_{2pq}$ and $\mathcal{C}^{6,\varsigma}_{2pq}$ are pairwise non-isomorphic, respectively. Assume $\mathcal{C}^{5,\xi_1}_{2pq} \cong \mathcal{C}^{5,\xi_2}_{2pq}$ for some $1 \le \xi_1, \xi_2 \le \frac{q-1}{2}$. The CI-property of $\mathcal{C}^{5,\xi}_{2pq}$ implies that there is an $\alpha \in \mathrm{Aut}(G)$ such that $\{c, ab^{\xi_1}, (ab^{\xi_1})^{-1}\}^{\alpha} = \{c, ab^{\xi_2}, (ab^{\xi_2})^{-1}\}$. Clearly, $c^{\alpha} = c$. By the argument in the first paragraph, $(ab^{\xi_1})^{\alpha} = ab^{\xi_2}$. Then $(a^{-1}b^{\xi_1})^{\alpha} = (cab^{\xi_1}c)^{\alpha} = cab^{\xi_2}c = a^{-1}b^{\xi_2}$ and $(a^2)^{\alpha} = (ab^{\xi_1}(a^{-1}b^{\xi_1})^{-1})^{\alpha} = ab^{\xi_2}(a^{-1}b^{\xi_2})^{-1} = a^2$, implying $a^{\alpha} = a$. It follows that $(b^{\xi_1})^{\alpha} = b^{\xi_2}$ and since $b^{-\xi_1}ab^{\xi_1} = a^{r^{\xi_1}}$, one may obtain $r^{\xi_1-\xi_2} = 1$ in $\mathbb{Z}_p$. Since r has order q in $\mathbb{Z}^*_p$ and $1 \le \xi_1, \xi_2 \le \frac{q-1}{2}$, one has $\xi_1 = \xi_2$. Thus, $\mathcal{C}^{5,\xi_1}_{2pq} \cong \mathcal{C}^{5,\xi_2}_{2pq}$ for some $1 \le \xi_1, \xi_2 \le \frac{q-1}{2}$ if and only if $\xi_1 = \xi_2$. Similarly, one may show that $\mathcal{C}^{6,\varsigma_1}_{2pq} \cong \mathcal{C}^{6,\varsigma_2}_{2pq}$ for some $1 \le \varsigma_1, \varsigma_2 \le \frac{q-1}{2}$ if and only if $\varsigma_1 = \varsigma_2$. □

The following theorem is the main result of this section.

Theorem 4.4. *Let $p > q$ be odd primes. A connected cubic Cayley graph of order $2pq$ is non-symmetric if and only if it is isomorphic to one of the following graphs: $\mathcal{C}^1_{2pq}$, $\mathcal{C}^{2,\lambda}_{2pq}(\lambda \in \Lambda)$, $\mathcal{C}^{3,\mu}_{2pq}(\mu \in \Theta)$, $\mathcal{C}^4_{2pq}$, $\mathcal{C}^{5,\xi}_{2pq}(1 \le \xi \le \frac{q-1}{2})$ and $\mathcal{C}^{6,\varsigma}_{2pq}(1 \le \varsigma \le \frac{q-1}{2}, q > 3, (p, q, \varsigma) \ne (11, 5, 2), (23, 11, 2))$, where Λ and Θ are given in Eq. (2).*

Proof. Let $X = \mathrm{Cay}(G, S)$ be a connected cubic non-symmetric Cayley graph on a group G of order $2pq$. Then $1 \notin S, S^{-1} = S$ and $\langle S\rangle = G$. Since X has valency 3, S contains an involution, say x. Let $A = \mathrm{Aut}(X)$ and A_1 the stabilizer of $1 \in G$ in A. To finish the proof, by Examples 4.1–4.3, it suffices to show that X is isomorphic to one of the graphs listed in the theorem. Recall that G is one of the groups $G_1(2pq)$, $G_2(2pq)$, $G_3(2pq)$, $G_4(2pq)$, $G_5(2pq)$ and $G_6(2pq)$ given in Eq. (1).

Let $G = G_1(2pq) = \langle a\rangle$. Then $S = \{x = a^{pq}, y, y^{-1}\}$, where y is an element of order pq or $2pq$. By Proposition 2.9, X is normal, and by Proposition 2.7, $A_1 = \mathrm{Aut}(G, S)$. Since X is non-symmetric, $\mathrm{Aut}(G_1(2pq), S) \le \mathbb{Z}_2$, and since the automorphism α of $G_1(2pq)$ induced by $a \mapsto a^{-1}$ fixes S setwise, $A_1 = \langle\alpha\rangle \cong \mathbb{Z}_2$ and $A = R(G_1(2pq)) \rtimes \langle\alpha\rangle$. It is easy to show that $\langle R(a^2), R(a^{pq})\alpha\rangle \cong G_2(2pq)$ acts regularly on $V(X)$, which implies that X is isomorphic to a Cayley graph on $G_2(2pq)$.

Let $G = G_2(2pq) = \langle a, b \mid a^{pq} = b^2 = 1, b^{-1}ab = a^{-1}\rangle$. Since all involutions of $G_2(2pq)$ are conjugate, one may let $x = b$. If $S = \{b, a^i, a^{-i}\}$ then $(i, pq) = 1$ because $\langle S\rangle = G$. Let α_i be the automorphism of $G_2(2pq)$ induced by $b \mapsto b$ and $a^i \mapsto a$. Then $S^{\alpha_i} = \{b, a, a^{-1}\}$, and hence $X \cong \mathcal{C}^1_{2pq}$. Now assume that S consists of three involutions. Then $S = \{b, ba^i, ba^j\}$ for some integers i, j. If one of i and j, say i, is coprime to pq then $S^{\alpha_i} = \{b, ba, ba^k\}$. If $(i, pq) \ne 1$ and $(j, pq) \ne 1$ then one of i and j is a multiple of p and the other is a multiple of q because $\langle S\rangle = G$, implying $(j - i, pq) = 1$. Let α be an automorphism of G mapping ba^i to b. Then $S^\alpha = \{b, b(a^{-i})^\alpha, b(a^{j-i})^\alpha\}$. Since $(j - i, pq) = 1$, one has $b(a^{j-i})^\alpha = ba^t$ for some t with $(t, pq) = 1$. Thus, $S^{\alpha\alpha_t} = \{b, ba, ba^k\}$. Without loss of any generality, one may assume $S = \{b, ba, ba^k\}$. Clearly, $k \ne 0, 1$. By Theorem 3.2, $X = \mathrm{Cay}(G, S)$ is normal, and by Proposition 2.7, $A_1 = \mathrm{Aut}(G, S)$. Recall that X is symmetric $(3 \mid |\mathrm{Aut}(G, S)|)$ if and only if $k^2 - k + 1 = 0$ in $\mathbb{Z}_{pq}$ if and only if $k \in -\mathcal{O}^3_{pq}$, where $\mathcal{O}^3_{pq}$ is the solution set of $x^3 + x + 1 = 0$ in $\mathbb{Z}_{pq}$. Thus, X is non-symmetric if and only if $k \notin -\mathcal{O}^3_{pq}$. By Example 4.1, if $k \in \{2^{-1}, \lambda, 1 - \lambda \mid \lambda \in \Lambda\}$ then $\mathrm{Aut}(X) \cong G \rtimes \mathbb{Z}_2$ and $X \cong \mathcal{C}^{2,\lambda}_{2pq}$ for some $\lambda \in \Lambda$; if $k \in \Theta = \mathbb{Z}_{pq} - \{\{0, 1, 2^{-1}, \lambda_1, \lambda_2, \lambda_3, 1 - \lambda_1, 1 - \lambda_2, 1 - \lambda_3\} \cup \{-\mathcal{O}^3_{pq}\}\}$ then $\mathrm{Aut}(X) = R(G)$ and $X \cong \mathcal{C}^{3,\mu}_{2pq}$.

Let $G = G_3(2pq)$. Since all involutions of $G_3(2pq)$ are conjugate and contained in the subgroup $\langle a, c\rangle$, by the connectivity of X, one may assume that $S = \{x = c, y, y^{-1}\}$, where y has order pq. Clearly, there exists an automorphism of $G_3(2pq)$ which fixes c and maps y to ab. It follows that $S \equiv \{c, ab, (ab)^{-1}\}$, and hence $X \cong \mathcal{C}^4_{2pq}$.

Let $G = G_4(2pq)$. By a similar argument to the above paragraph, one may let $S = \{c, ab, (ab)^{-1}\}$. By Theorem 3.2, $X = \mathrm{Cay}(G, S)$ is normal, and hence $A_1 = \mathrm{Aut}(G, S)$. It is easy to check that $\mathrm{Aut}(G, S) = \langle\alpha\rangle \cong \mathbb{Z}_2$, where α is the automorphism of G induced by $c \mapsto c$, $a \mapsto a^{-1}$ and $b \mapsto b^{-1}$. Let $H = \langle R(a), R(b), R(c)\alpha\rangle$. Direct calculation shows that $R(a)^{R(c)\alpha} = R(a)^{-1}$, $R(b)^{R(c)\alpha} = R(b)$ and $R(c)^\alpha = R(c)$. It follows that $H \cong G_3(2pq)$. If $H_1 \ne 1$ then $H_1 = \mathrm{Aut}(G, S)$ and then $\alpha \in H$, forcing $H = A$, a contradiction. Thus, H is regular on $V(X)$ and hence X is also a Cayley graph on $G_3(2pq)$, which is discussed in the previous paragraph.

Let $G = G_5(2pq)$. Then c is in the center of G. Since G has no elements of order pq, there is no connected cubic Cayley graph on $G_5(2pq)$.

Let $G = G_6(2pq)$. Since all involutions of G are conjugate and contained in the subgroup $\langle a, c\rangle$, the connectivity of X implies that $S = \{c, y, y^{-1}\}$, where y has order q or $2q$. If y has order q then $y = a^ib^k$ with $1 \le i \le p - 1$ and $1 \le k \le q - 1$. Let α_i be the automorphism of G induced by $a^i \mapsto a$, $b \mapsto b$ and $c \mapsto c$. Then $S^{\alpha_i} = \{c, ab^k, (ab^k)^{-1}\}$. One may assume $1 \le k \le \frac{q-1}{2}$ because the map β_1 defined by $a \mapsto a^{-r^{q-k}}$, $b \mapsto b$, $c \mapsto c$ induces an automorphism of G and $(ab^k)^{\beta_1} = a^{-r^{q-k}}b^k = b^ka^{-1} = (ab^{-k})^{-1}$. Thus, $X \cong \mathcal{C}^{5,\xi}_{2pq}$, $1 \le \xi \le \frac{q-1}{2}$. If y has order $2q$ then $y = a^ib^kc$ with $1 \le i \le p - 1$ and $1 \le k \le q - 1$. Clearly, $S^{\alpha_i} = \{c, ab^kc, (ab^kc)^{-1}\}$, and one may assume $1 \le k \le \frac{q-1}{2}$ because the map β_2 defined by $a \mapsto a^{r^{q-k}}$, $b \mapsto b$, $c \mapsto c$ induces an automorphism of $G_6(2pq)$ and $(ab^kc)^{\beta_2} = (ab^{-k}c)^{-1}$. It follows that $X \cong \mathcal{C}^{6,\varsigma}_{2pq}$, $1 \le \varsigma \le \frac{q-1}{2}$. Since X is non-symmetric, by Example 4.3, $p > q > 3$ and $(p, q, \varsigma) \ne (11, 5, 1), (23, 11, 3)$. $\square$

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